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Automorphisms of elliptic surfaces, inducing the identity in cohomology ☆

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ABSTRACT

We give a numerical classification for pairs (S, G) consisting of complex nonsingular projective surfaces S of Kodaira dimension one and subgroups G of automorphisms of S inducing trivial actions on $H^2(S, \mathbb{Q})$.

0. Introduction

Let S be a complex nonsingular projective surface, and let $G \subset \operatorname{Aut} S$ be the subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$. It is known that, for K3 surfaces S, G is trivial (cf. [BR]). For Enriques surfaces S, Mukai and Namikawa [Mu-Na] showed that G is trivial or a cyclic group of order 2 or 4, and classified pairs (S, G) with G non-trivial. For surfaces S with base-point-free canonical linear system $|K_S|$, Peters [Pet1] proved the pairs (S, G) satisfy either $K_S^2 = 8\chi(\mathcal{O}_S)$ and $|G|$ is a power of 2, or $K_S^2 = 9\chi(\mathcal{O}_S)$ and $|G|$ is a power of 3. Recently, Chen [Ch] gave further information about this group. For surfaces S with a pencil of genus 2, we proved $|G| \leqslant 2$, and we classified pairs (S, G) with $|G| = 2$ [Ca2,Ca3].

In this note, we consider surfaces S of Kodaira dimension one. Roughly speaking, our main result is the following:

Theorem 0.1. *(See Theorems 1.8 and 3.1.) Let S be a complex nonsingular projective elliptic surface of Kodaira dimension one, and $G \subset \operatorname{Aut} S$ the subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$.*

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(i) *If* $\chi(\mathcal{O}_S) \geqslant 2$, *then* G *is trivial*; *i.e., the group* Aut S *of automorphisms of* S *induces faithful action on* $H^2(S, \mathbb{Q})$.
(ii) *if* $\chi(\mathcal{O}_S) = 1$ *and* $p_g(S) > 0$, *then* G *is trivial or a cyclic group of order* 2. *Moreover, if* G *not trivial, the generator of* G *induces* $-\mathrm{id}$ *on* $H^1(S, \mathbb{Q})$.
(iii) *If* $\chi(\mathcal{O}_S) = 0$, $p_g(S) \geqslant 3$, *and* S *has no non-trivial holomorphic vector fields, then* G *is isomorphic to one of the following six groups*: $\mathbb{Z}_n$ $(n \leqslant 3)$, $\mathbb{Z}_2^{\oplus 2}$, Sym(3), $\mathbb{Z}_2^{\oplus 3}$.

Remark 0.2. Pairs (S, G) in (ii) of Theorem 0.1 with $|G| = 2$ are obtained in the way of Proposition-Construction 4.1 and do really occur (Example 4.2). Examples of pairs (S, G) in (iii) of Theorem 0.1 with $|G| \geqslant 6$ are explicitly constructed (Examples 4.4 and 4.5).

As a consequence of Theorem 0.1, we have a result of Peters [Pet2]:

Corollary 0.3. *Let* S *be as in Theorem* 0.1. *If* $\chi(\mathcal{O}_S) > 0$ *and* $p_g(S) > 0$, *then the group* Aut S *of automorphisms of* S *induces faithful action on the cohomology ring* $H^*(S, \mathbb{Q})$.

We shall use Kodaira's notation for elliptic fibers; for convenience, we identity ${}_1\mathrm{I}_b$ with I_b. We denote by $e(X)$ the Euler topological characteristic of a topological space X. The symbols $\mathbb{Z}_n$ and Sym(n) denote respectively the cyclic group of order n and the symmetric group on n letters.

1. $\chi(\mathcal{O}_S) \geqslant 3$

1.1. Let S be a complex nonsingular minimal projective surface with Kodaira dimension one and $h^0(\mathcal{T}_S) = 0$, where $\mathcal{T}_S$ is the tangent sheaf of S. Let

$$f : S \to C$$

be the Iitaka fibration (i.e., the morphism associated to the linear system $|mK_S|$ for m sufficiently large), and $m_1 D_1, \ldots, m_k D_k$ $(k \geqslant 0)$ all the multiple fibers of f, with their multiplicities attached. We have the canonical bundle formula for f (see e.g. [BHPV])

$$\omega_S = f^*(\mathcal{L} \otimes \omega_C) \otimes \mathcal{O}_S\left(\sum_{i=1}^{k}(m_i - 1)D_i\right), \tag{1.1.1}$$

where $\mathcal{L} = f_*(\omega_S \otimes f^*\omega_C^{-1})$ is the dualizing sheaf of f. We have that $\mathcal{L}$ is invertible and $\deg \mathcal{L} = \chi(\mathcal{O}_S) \geqslant 0$.

Let $G \subset \mathrm{Aut}\, S$ be the subgroup of automorphisms of S, inducing trivial actions on $H^2(S, \mathbb{Q})$. One has that G is a finite group (cf. e.g. [Pet1, Lemma 1]). Clearly f is preserved under the action of G; that is, for each $\sigma \in G$, there is an automorphism $\bar{\sigma} \in \mathrm{Aut}\, C$ such that $\bar{\sigma} \circ f = f \circ \sigma$. The map $G \to \mathrm{Aut}\, C$, sending σ to $\bar{\sigma}$, is a homomorphism. Denote by H its kernel and Q its image. We have an exact sequence of groups

$$1 \to H \to G \to Q \to 1.$$

Let F be a general fiber of f. Then H is a subgroup of Aut F.

Lemma 1.2. *Notations are as above. We have*

(i) *If* $p_g(S) > 0$ *then* H *consists of translations of* F;
(ii) *If* $p_g(S) \geqslant 2$ *then* $|Q| \leqslant 1 + \frac{g(C)}{\chi(\mathcal{O}_S) - 2 + g(C)}$.

Proof. Since $p_g(S/H) = p_g(S)$, the induced fibration $S/H \to C$ must be elliptic if $p_g(S) > 0$. This implies (i).

If $p_g(S) \geqslant 2$, by (1.1.1), we have that the moving part of $|K_S|$ is base-point-free and the canonical map ϕ_S of S factors as

$$\phi_S = \varphi \circ f : S \to C \to \Sigma := \operatorname{Im} \phi_S \subset \mathbb{P}^{p_g(S)-1},$$

where φ is the map associated to $\mathcal{L} \otimes \omega_C$. From $\deg \varphi \deg \Sigma \leqslant \deg(\mathcal{L} \otimes \omega_C)$, we get

$$\deg \varphi \leqslant \frac{\chi(\mathcal{O}_S) + 2g(C) - 2}{p_g(S) - 1} = 1 + \frac{g(C)}{\chi(\mathcal{O}_S) - 2 + g(C)},$$

where the right equality follows from Lemma 1.3 below.

Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \mathbb{C})$, G acts trivially on $H^0(S, \omega_S)$. This implies that G induces trivial actions on Σ. So φ factors through the quotient map

$$\varphi = \alpha \circ q : C \xrightarrow{q} C/Q \xrightarrow{\alpha} \Sigma.$$

Thus

$$|Q| \leqslant \deg \varphi.$$

By the above two inequalities, we get (ii). $\square$

Lemma 1.3. *Let $f : S \to C$ be as in* 1.1. *We have $q(S) = g(C)$.*

Proof. Otherwise, by the Leray spectral sequence,

$$h^0\big(R^1 f_* \mathcal{O}_S\big) = q(S) - g(C) > 0.$$

Since $R^1 f_* \mathcal{O}_S \simeq \mathcal{L}^{-1}$ (the relative duality) and $\deg \mathcal{L} = \chi(\mathcal{O}_S) \geqslant 0$, we have that $\mathcal{L} \simeq \mathcal{O}_C$. Then $c_2(S) = 12\chi(\mathcal{O}_S) = 0$. So $f : S \to C$ is a quasi-bundle; that is, the only singular fibers of f are of type ${}_m\mathrm{I}_0$. Then the relative tangent sheaf $\mathcal{T}_{S/C}$ is isomorphic to $\mathcal{O}_S(-K_S + f^*K_C + \sum_{i=1}^k (m_i - 1)D_i)$ (see e.g. [Se, 3.1]). By (1.1.1), we have $K_S \equiv f^*K_C + \sum_{i=1}^k (m_i - 1)D_i$. So $\mathcal{T}_{S/C} \simeq \mathcal{O}_S$. From the exact sequence of sheaves

$$0 \to \mathcal{T}_{S/C} \to \mathcal{T}_S \to f^*\mathcal{T}_C,$$

we have $h^0(\mathcal{T}_S) > 0$, which contracts the assumption $h^0(\mathcal{T}_S) = 0$. $\square$

Here we insert a lemma that will be used in the sequel. Recall that, for an automorphism σ of finite order r of surface S, we say that the action of σ at an isolated σ-fixed point $x \in S$ is of weight $\frac{1}{r}(a, b)$ $(0\langle a, b\rangle r)$ if the induced linear action of σ on $T_x S \simeq \mathbb{C}^2$ can be given with respect to suitable basis (v_1, v_2) by

$$\sigma(v_1, v_2) = \big(e^{2\pi a\sqrt{-1}/r} v_1, e^{2\pi b\sqrt{-1}/r} v_2\big).$$

Lemma 1.4. *Let $f : S \to C$ be a relatively minimal fibration of genus $g \geqslant 1$, and σ an automorphism of finite order r of S with $f \circ \sigma = f$. Assume that $p \in S$ is an isolated fixed point of σ. Then*

(i) *$F' := f^*(f(p))$ is singular at p;*
(ii) *If moreover $\operatorname{mult}_p F' = 2$ and r is an odd prime, then p is a node of F', and the action of σ at p is of weight $\frac{1}{r}(1, r-1)$.*

Proof. Locally near p, the action of σ on S can be given with respect to suitable local coordinates (x, y) of S around p by

$$\sigma(x) = \zeta x, \qquad \sigma(y) = \zeta^a y$$

with ζ a primitive r-th root of unity and integer a satisfying $1 \leqslant a < r$. Let t be a local coordinate of C around $f(p)$. Then within an analytic neighborhood of p,

$$f^*t = c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2 + \text{higher order terms}$$

for some complex numbers $c_1, \ldots, c_5$. Since σ induces the trivial action on C, we have that f^*t is σ-invariant. So

$$c_1x + c_2y = c_1\zeta x + c_2\zeta^a y,$$
$$c_3x^2 + c_4xy + c_5y^2 = c_3\zeta^2x^2 + c_4\zeta^{1+a}xy + c_5\zeta^{2a}y^2.$$

From the first equality we get $c_1 = c_2 = 0$. This implies (i). Under the assumption of (ii), from the second equality we have $c_3 = c_5 = 0$, $c_4 \neq 0$ and $a = r - 1$, and hence (ii) follows. $\square$

Lemma 1.5. *Let $f : S \to C$, H be as in 1.1, and F' a fiber of f. Assume that $p_g(S) > 0$ and that H is not trivial. Let $\sigma \in H$ be an element of prime order r. Then the possible types for F' and the action of σ on F' are as follows.*

(i) *F' is of type ${}_mI_0$, $r|m$, and F'_{red} is σ-fixed;*
(ii) *F' is of type ${}_mI_0$ ($m \geqslant 1$), and σ acts on F'_{red} as a translation;*
(iii) *F' is of type ${}_mI_b$ ($m \geqslant 1$, $b > 0$), and the action of σ at nodes of F'_{red} is of weight $\frac{1}{r}(1, r-1)$.*

Proof. We repeatedly use three simple observations:

(1.5.1) Each irreducible component of F' is σ-invariant [Ca1, 3.1];
(1.5.2) Let $\Theta < F'$ be an irreducible component. If $\text{mult}_\Theta F' = 1$ then Θ is not σ-fixed [Ca0, 2.3.1];
(1.5.3) σ has no horizontal fixed curves (by (i) of Lemma 1.2).

If F' is not of type ${}_m\mathrm{I}_b$, then by Kodaira's classification of singular fibers, there exists an irreducible component Θ of F' such that $\text{mult}_\Theta F' = 1$ and either F' is irreducible (type II) or Θ meets $\text{supp}(F' - \Theta)$ in one point, say q. By (1.5.1) and (1.5.2), $\sigma_{|\Theta}$ is a non-trivial automorphism of Θ. Besides q or the cusp when F' is irreducible, σ has a fixed point, say q', on Θ. By (1.5.3), q' is an isolated σ-fixed point. Since q' is a smooth point of F', we get a contradiction by (i) of Lemma 1.4.

Now we can assume that F' is of type ${}_m\mathrm{I}_b$.

Case 1. $m = 1$. If $b = 0$, by (1.5.2) and (i) of Lemma 1.4, $\sigma_{|F'}$ is a translation. If $b > 0$, by (1.5.1)–(1.5.3), the nodes of F' are isolated σ-fixed. When $r > 2$, by (ii) of Lemma 1.4, the action of σ at nodes of F' is of type $\frac{1}{r}(1, r-1)$. So in this case the action of σ on F' is in (ii)–(iii) of Lemma 1.5 with $m = 1$.

Case 2. $m > 1$. Let $\Delta \subset C$ be a unit disk around $t = f(F')$ with coordinate z, such that f is smooth over $\Delta \setminus \{t\}$. We have a commutative diagram

$$\begin{tikzcd}
\bar{S}_\Delta \arrow[r, "\rho"] \arrow[dr, "\bar{f}_\Delta"'] & \Delta \times_\Delta S_\Delta \arrow[r, "\tilde{\mu}"] \arrow[d] & S_\Delta := f^{-1}\Delta \arrow[d, "f_\Delta := f_{|S_\Delta}"] \\
& \Delta \arrow[r, "\mu"] & \Delta
\end{tikzcd}$$

where $\mu : \Delta \to \Delta$ is the map $z \mapsto z^m$, ρ is the normalization of the fiber product $\Delta \times_\Delta S_\Delta$ with respect to μ, and $\bar{f}_\Delta$ is the induced fibration. Then the central fiber $\bar{F}'$ of $\bar{f}_\Delta$ is of type I_{mb}. Let $\bar{\sigma}$ be the automorphism of $\bar{S}_\Delta$ induced by σ, and β the automorphism of $\bar{S}_\Delta$ corresponding to $\tilde{\mu} \circ \rho$. We have $\bar{\sigma} \circ \beta = \beta \circ \bar{\sigma}$.

If $b = 0$, then $\beta_{|\bar{F}'}$ is a translation of order m, and as in Case 1, $\bar{\sigma}_{|\bar{F}'}$ is also a translation. So the action of σ on F' is as in (i)–(ii) of Lemma 1.5.

Now we assume that $b \geqslant 1$. First we show that the nodes of F'_{red} are isolated σ-fixed. This is true if $b = 1$. When $b \geqslant 2$, F'_{red} is a cycle of smooth rational curves. We label irreducible components $\Theta_0, \Theta_1, \dots, \Theta_{b-1}$ of F' with subscripts mod b so that $\Theta_i \Theta_{i+1} = 1$. Replacing Δ with a smaller disk if necessary, we have that there exists an automorphism α of S_Δ, of order mb, satisfying the following conditions:

(a) $f_\Delta \circ \alpha = f_\Delta$;
(b) α acts on a general fiber F of f_Δ as a translation;
(c) $\alpha(\Theta_i) = \Theta_{i+1}$ for $i = 0, 1, \dots, b-1$.

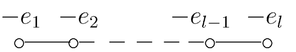
Indeed, by Kodaira's construction [K, p. 600], there exists an automorphism $\bar{\alpha}$ of $\bar{S}_\Delta$, of order mb, satisfying conditions (a)–(c) as above (with $\bar{\alpha}$ instead of α, $\bar{f}_\Delta$ instead of f_Δ, and so on) and $\bar{\alpha} \circ \beta = \beta \circ \bar{\alpha}$. So there exists an automorphism α of S_Δ with $\tilde{\mu} \circ \rho \circ \bar{\alpha} = \alpha \circ \tilde{\mu} \circ \rho$, which is easily checked to have the desired properties.

Let $\sigma_\Delta = \sigma_{|S_\Delta}$. By (b), we have that σ_Δ communicates with α. Combining this fact with (c), we infer that Θ_i is not σ-fixed for all i. Hence the nodes of F'_{red} are isolated σ-fixed.

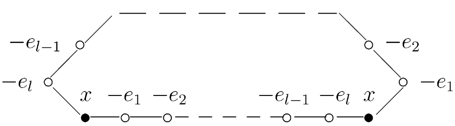
Now by (c), the action of σ on the nodes of F'_{red} is of same weight, say, $\frac{1}{r}(1, v)$ for some $1 \leqslant v < r$. Let

$$[e_1, e_2, \dots, e_l] = e_1 - \cfrac{1}{e_2 - \cfrac{1}{e_3 - \cdots}}$$

be the Hirzebruch–Jung continued fraction of $\frac{r}{v}$. Let $\delta : T \to S_\Delta/\sigma_\Delta$ be the minimal desingularization and $h : T \to \Delta$ the associated fibration. Then



is the dual graph of the fiber of δ over each cyclic quotient singularity of S_Δ/σ_Δ, and the dual graph of h^*t is a cycle



Here $\bullet$ indicates the strict transform of the image of irreducible components of F' in S/σ, and x is its self-intersection number. If $x \neq -1$, then h is relatively minimal, and by Kodaira's classification for singular elliptic fibers, we have $-x = e_1 = \dots = e_l = 2$. In this case, $v = r - 1$, and the action of σ on F' is as in (iii) of Lemma 1.5 with $m > 1$.

Now we suppose that $x = -1$. Note that the Kodaira dimension of S is one, e_1 and e_l can not both be equal to 2. Since the singular fiber of the relatively minimal model of h is of type ${}_{m'}\mathrm{I}_{b'}$, we have that $(e_1, \dots, e_l) \in \mathcal{E}$, where $\mathcal{E}$ is the set of tuples of positive integers such that

(i) $(4) \in \mathcal{E}$, and for $i = 0, 1, \dots,$ $(3, \overbrace{2, 2, \dots, 2}^{i}, 3) \in \mathcal{E}$;
(ii) if $(a_1, \dots, a_j) \in \mathcal{E}$, then so are $(2, a_1, \dots, a_{j-1}, a_j + 1)$ and $(a_1 + 1, a_2 \dots, a_j, 2)$;

(iii) each $(a_1,\dots,a_j)\in\mathcal{E}$ can be obtained by starting with one of (i) and iterating the steps described in (ii).

Since $[e_1,\dots,e_l]=\frac{r}{v}$ and r is a prime, we get a contradiction by showing that, for each $(a_1,\dots,a_j)\in\mathcal{E}$, $[a_1,\dots,a_j]=\frac{ks^2}{uks-1}$ for some $k\geqslant 1$, $s\geqslant 2$ and $1\leqslant u<s$. Indeed, we have $[4]=\frac{2^2}{1}$; for $i\geqslant 0$, $[3,\overbrace{2,2,\dots,2}^{i},3]=\frac{(i+2)2^2}{2(i+2)-1}$. Inductively, if $[a_1,\dots,a_j]=\frac{ks^2}{uks-1}$ for some $k\geqslant 1$, $s\geqslant 2$ and $1\leqslant u<s$, then by an elementary calculation, we have $[2,a_1,\dots,a_{j-1},a_j+1]=\frac{k(2s-u)^2}{sk(2s-u)-1}$ and $[a_1+1,a_2\dots,a_j,2]=\frac{k(s+u)^2}{uk(s+u)-1}$. This finishes the proof of the induction. □

1.6. The equivariant signature formula

For the reader's convenience, we recall the equivariant signature formula for a finite Abelian group action on a surface; see [HZ] for more details.

Let S be a complex nonsingular projective surface, and G a finite Abelian subgroup of automorphisms of S. For each subvariety $Y\subset S$, let

$$G_Y=\bigl\{\sigma\in G \bigm| Y \text{ is } \sigma\text{-fixed, i.e., } \sigma(y)=y \text{ for every point } y\in D\bigr\}.$$

For each point $x\subset S$, since G_x is Abelian, there exist characters χ_1 and χ_2 of G_x such that the induced linear action of G_x on the tangent space T_xS of S at x can be given with respect to suitable coordinates (v_1,v_2) by

$$\sigma(v_1,v_2)=\bigl(\chi_1(\sigma)v_1,\chi_2(\sigma)v_2\bigr).$$

Let H be the image of $\chi_1\times\chi_2: G_x\to\mathbb{C}^*\times\mathbb{C}^*$, and H_1 (resp. H_2) the intersection of H with $\mathbb{C}^*\times\{1\}$ (resp. $\{1\}\times\mathbb{C}^*$). Then as a subgroup of $\mathbb{C}^*/H_1\times\mathbb{C}^*/H_2\simeq\mathbb{C}^*\times\mathbb{C}^*$, $\bar{H}:=H/(H_1\times H_2)$ is of the form

$$\bigl\{\bigl(e^{2\pi ik/n_x},e^{2\pi ikq_x/n_x}\bigr) \bigm| 0\leqslant k<n_x\bigr\}\quad\bigl(n_x=|\bar{H}|\bigr)$$

with integer q_x satisfying $(q_x,n_x)=1$. Define the defect of x (with respect to G) by

$$\operatorname{def}_x=-4|G_x|\sum_{k=1}^{n_x-1}\left(\left(\frac{k}{n_x}\right)\right)\left(\left(\frac{kq_x}{n_x}\right)\right),$$

where for $r\in\mathbb{R}$, $((r))=r-[r]-\frac{1}{2}$ (resp. $=0$) if $r\in\mathbb{R}\setminus\mathbb{Z}$ (resp. $r\in\mathbb{Z}$) (here $[r]$ is the greatest integer $\leqslant r$). We have

$$|G|\operatorname{Sign}(S/G)=\operatorname{Sign}(S)+\sum_{\text{smooth curve } D\subset S}\frac{|G_D|^2-1}{3}D^2+\sum_{x\in S}\operatorname{def}_x. \tag{1.6.1}$$

Proposition 1.7. *Let S, H be as in* (1.1)*. Assume that $p_g(S)>0$.*

(i) *If $\chi(\mathcal{O}_S)>0$, then H is trivial.*
(ii) *If $\chi(\mathcal{O}_S)=0$, then H is exactly the subgroup H' of automorphisms ς of S satisfying the following conditions*: 1) $f\circ\varsigma=f$, *and* 2) $\varsigma_{|F}$ *is a translation for a general fiber F, and it is isomorphic to one of the following four groups*: $\mathbb{Z}_n$ $(n\leqslant 3)$, $\mathbb{Z}_2^{\oplus 2}$.

Proof. Assume that H is not trivial; let $\sigma \in H$ be an element of prime order r. By Lemma 1.5, σ has no fixed curves with non-zero self-intersection number and each isolated σ-fixed point $p \in S$ is of type $\frac{1}{r}(1, r-1)$. We may apply the equivariant signature formula to σ. We have (notations as in 1.6)

$$\begin{aligned}\operatorname{def}_p &= -4r\sum_{k=1}^{r-1}\left(\left(\frac{k}{r}\right)\right)\left(\left(\frac{(r-1)k}{r}\right)\right) = -4r\sum_{k=1}^{r-1}\left(\frac{k}{r}-\frac{1}{2}\right)\left(\frac{-k}{r}+\frac{1}{2}\right)\\ &= \frac{1}{r}\sum_{k=1}^{r-1}(2k-r)^2 = \frac{(r-1)(r-2)}{3}.\end{aligned}$$

By (1.6.1), we have

$$(r-1)\bigl(K_S^2 - 8\chi(\mathcal{O}_S)\bigr) = (r-1)\operatorname{Sign}(S) = \frac{(r-1)(r-2)}{3}k \geqslant 0, \tag{1.7.1}$$

where k is the number of isolated σ-fixed points.

If $\chi(\mathcal{O}_S) > 0$, we get a contradiction by (1.7.1). This proves (i).

Now we assume $\chi(\mathcal{O}_S) = 0$. Since $e(S) = 12\chi(\mathcal{O}_S) = 0$, $f: S \to C$ is a quasi-elliptic bundle. So it admits a very concrete description, i.e., there exists a finite group Δ acting faithfully on a smooth fiber F of f and on some smooth curve $\tilde{C}$ such that f is equivalent to the fiber surface

$$p : (F \times \tilde{C})/\Delta \to \tilde{C}/\Delta,$$

where the action of Δ on $F \times \tilde{C}$ is the diagonal free action and p is the projection to the second factor (cf. [Se]). For each $\varsigma \in H'$, ς induces an automorphism of $\tilde{C} \times_C S$, which is of the form $\mathrm{id}_{\tilde{C}} \times t_x$ under the identification of $\tilde{C} \times_C S$ with $\tilde{C} \times F$, where t_x is the translation of F by x for some $x \in F$. So ς has no isolated fixed points, and hence one checks easily that $T := S/\varsigma \to C$ is a quasi-elliptic bundle; in particular, $e(T) = 0$. From $g(C) = q(S) \geqslant q(T) \geqslant g(C)$, we get $q(S) = q(T)$. So $b_2(T) = 4q(T) - 2 = 4q(S) - 2 = b_2(S)$. This implies the induced action of ς on $H^2(S, \mathbb{Q})$ is trivial. Hence $H' \subset H$. On the other hand, by (i) of Lemma 1.2, $H \subset H'$. So $H = H'$.

To compute H', we describe explicitly the diagonal action.

Write $F = \mathbb{C}/(\mathbb{Z} + \xi\mathbb{Z})$ for some $\xi \in \mathbb{C} \setminus \mathbb{R}$, and identify Δ as a subgroup $T \rtimes A$ (a semi-direct product) of $\operatorname{Aut} F$, where T is a group of translations and A is a subgroup preserving the group structure. Since S has no non-trivial holomorphic vector fields, A is not trivial.

Let $\psi : \Delta \to \operatorname{Aut}\tilde{C}$ be the homomorphism of groups such that the diagonal action of Δ on $\tilde{C} \times F$ is given by $\alpha(x, y) = (\psi(\alpha)(x), \alpha(y))$, for $\alpha \in \Delta$, $(x, y) \in \tilde{C} \times F$. Since the diagonal action is faithful, ψ is injective. Let Γ_ψ be the graph of ψ. Then Γ_ψ is a subgroup of $\operatorname{Aut}(\tilde{C} \times F)$, and by the argument in the proof of $H = H'$, we have $H' \simeq (\mathrm{N}(\Gamma_\psi) \cap T_f)/\Gamma_\psi \cap T_f$, where $\mathrm{N}(\Gamma_\psi)$ is the normalizer of Γ_ψ in $\operatorname{Aut}(\tilde{C} \times F)$ and $T_f = \{\mathrm{id}_{\tilde{C}} \times t_a \mid a \in F\}$. Since ψ is injective, we have $\Gamma_\psi \cap T_f$ is trivial, and so $H' \simeq \mathrm{N}(\Gamma_\psi) \cap T_f$.

Let $z \in F$ such that $\beta := \mathrm{id}_{\tilde{C}} \times t_z \in \mathrm{N}(\Gamma_\psi) \cap T_f$. For each $\alpha \in \Delta$, we have $\psi(\alpha) \times t_z\alpha t_{-z} = \beta(\psi(\alpha) \times \alpha)\beta^{-1} \in \Gamma_\psi$. Since ψ is injective, we must have $t_z\alpha t_{-z} = \alpha$. This implies that z is A-invariant. So $\mathrm{N}(\Gamma_\psi) \cap T_f$ is isomorphic to the group of translations of A-invariant elements. Now by a direct computation (cf. e.g. [Be, p. 87]), H' is given up to isomorphism as in Table 1, where $\rho = e^{\frac{2\pi\sqrt{-1}}{3}}$.

This finishes the proof of (ii). $\square$

Theorem 1.8. *Let S be a complex nonsingular projective elliptic surface. If $\chi(\mathcal{O}_S) \geqslant 3$, then the group* Aut *$S$ of automorphisms of S induces faithful action on $H^2(S, \mathbb{Q})$.*

Table 1

A	ξ	The action of A on F	A-invariant elements	H'
$\mathbb{Z}_2$	arbitrary	$z \mapsto -z$	Points of order 2	$\mathbb{Z}_2^{\oplus 2}$
$\mathbb{Z}_3$	ρ	$z \mapsto \rho z$	$0,\ \pm\frac{1-\rho}{3}$	$\mathbb{Z}_3$
$\mathbb{Z}_4$	$\sqrt{-1}$	$z \mapsto \sqrt{-1}z$	$0,\ \frac{1+\sqrt{-1}}{2}$	$\mathbb{Z}_2$
$\mathbb{Z}_6$	ρ	$z \mapsto -\rho z$	0	1

Proof. Let Q and H be as in (1.1). If $\chi(\mathcal{O}_S) \geqslant 3$, then Q is trivial by (ii) of Lemma 1.2 and H is trivial by (1.7). $\square$

2. Classification of automorphisms of *G* of higher order

Proposition 2.1. *Let $f : S \to C$, G and Q be as in* (1.1)*. Assume that Q is not trivial. Let $\sigma \in G$ such that its image $\bar{\sigma} \in Q$ is an element of prime order, say r. Then*

(i) *The order* $\mathrm{o}(\sigma)$ *of $\sigma \in G$ equals r*;
(ii) $r = 2$.

We begin with the following lemma.

Lemma 2.2. *Notations as above. Let $h : T \to B := C/\bar{\sigma}$ be the relatively minimal fibration of* (*a smooth model of*) *the elliptic fiber space $S/\sigma \to C/\bar{\sigma}$. Then*

(i) $p_g(T) = p_g(S)$;
(ii) $q(T) = g(B)$;
(iii) $g(B) = 0$ (*resp.* $\leqslant 1$) *if* $\chi(\mathcal{O}_S) = 1$, *or* 2 (*resp.* 0);
(iv) $\mathrm{Trace}(\sigma|H^1(S,\mathbb{C})) = \frac{2rg(B)-2g(C)}{r-1}$.

Proof. (i) Since $H^0(\omega_S)$ is a direct factor of $H^2(S,\mathbb{C})$, (i) follows from the assumption that the induced action of σ on $H^2(S,\mathbb{Q})$ is trivial.

(ii) and (iv) By Lemma 1.3, f induces an isomorphism $f^* : H^1(C,\mathbb{C}) \to H^1(S,\mathbb{C})$, which is compatible with the induced action of σ since $\bar{\sigma} \circ f = f \circ \sigma$, we have that

$$q(T) = \dim H^0\big(S, \Omega_S^1\big)^{\sigma} = \dim H^0\big(C, \Omega_C^1\big)^{\bar{\sigma}} = g(B),$$

and

$$\mathrm{Trace}\big(\sigma \bigm| H^1(S,\mathbb{C})\big) = \mathrm{Trace}\big(\bar{\alpha} \bigm| H^1(C,\mathbb{C})\big) = \frac{2rg(B) - 2g(C)}{r-1},$$

where the last equality follows by applying the topological Lefschetz formula to $\bar{\sigma}$.

(iii) If $p_g(S) = 1$, then $g(C) = 2 - \chi(\mathcal{O}_S)$. So (iii) is trivially true except the case that $g(C) = \chi(\mathcal{O}_S) = 1$ and $\bar{\sigma}$ is a translation of C. We show that the exceptional case does not occur. Otherwise, we have that $S \to S/\sigma$ is étale, and hence $1 = \chi(\mathcal{O}_S) = \mathrm{o}(\sigma)\chi(\mathcal{O}_{S/\sigma})$, a contradiction.

Now we assume $p_g(S) \geqslant 2$. Notations as in the proof of Lemma 1.2. We have that

$$2\deg\alpha\big(\chi(\mathcal{O}_S) + g(C) - 2\big) \leqslant \mathrm{o}(\sigma)\deg\alpha \cdot \deg\Sigma \leqslant \chi(\mathcal{O}_S) + 2g(C) - 2.$$

If $\chi(\mathcal{O}_S) = 1$, or 2, the above inequality implies that $\deg\alpha = 1$ and $\deg\Sigma = \operatorname{codim}\Sigma + 1$. So Q is generated by $\bar{\sigma}$, C/Q is isomorphic to Σ, and Σ is a smooth rational curve.

If $\chi(\mathcal{O}_S) = 0$, the above inequality implies that either $\deg\alpha = 1$ and $\deg\Sigma = \operatorname{codim}\Sigma + 1$ or $\operatorname{codim}\Sigma + 2$, or $\deg\alpha = 2$ and $g(C) = 3$. In the former case, Q is generated by $\bar{\sigma}$, C/Q is isomorphic to Σ, and Σ is either a smooth rational curve or an elliptic curve. In the latter case, suppose that $g(B) \geqslant 2$. Then $\bar{\sigma}$ must be a fixed-point-free involution of C, and hence $S \to S/\sigma$ is étale. So $\chi(\mathcal{O}_{S/\sigma}) = 0$. On the other hand, by (i)–(ii) of Lemma 2.2, we have $p_g(S/\sigma) = 2$ and $q(S/\sigma) = 2$. So $\chi(\mathcal{O}_{S/\sigma}) = 1$, a contradiction. □

Proof of Proposition 2.1. (i) If $\chi(\mathcal{O}_S) \geqslant 1$, by (i) of Proposition 1.7, $G \simeq Q$, and the lemma is trivially true. Now assume $\chi(\mathcal{O}_S) = 0$. Since the image of σ^r in Q is trivial, we have that $\sigma^r \in H$. Suppose that $\mathrm{o}(\sigma) \neq r$. Then $\sigma^r \neq \mathrm{id}$. By the proof of Proposition 1.7(ii), σ^r, and hence σ, has no isolated fixed points. So S/σ is smooth and $S/\sigma \to B$ is a quasi-elliptic bundle. This implies $\chi(\mathcal{O}_{S/\sigma}) = \frac{1}{12}e(S/\sigma) = 0$. By (i)–(ii) of Lemma 2.2, we have $\chi(\mathcal{O}_{S/\sigma}) = 1 - g(B) + (g(C) - 1) = g(C) - g(B)$. So $g(C) = g(B)$ and hence $g(C) = 1$. This is a contradiction since $g(C) = q(S) = p_g(S) + 1 \geqslant 2$.

(ii) We show that $r \geqslant 3$ does not occur. First we suppose that $r \geqslant 5$. If $p_g(S) \geqslant 2$, by the proof of Lemma 1.2, we have that $|Q| \leqslant 4$. So we may assume that $p_g(S) = 1$. We distinguish three cases according to $\chi(\mathcal{O}_S)$:

Case 1. $\chi(\mathcal{O}_S) = 0$. Since f is quasi-elliptic, by the fact that the order of an automorphism of an elliptic curve with fixed points is 2, 3, 4, or 6, we have that, for each $\bar{\sigma}$-fixed point $t \in C$, the restriction of σ to f^*t is either identity or a translation. This implies that $S/\sigma \to B$ is quasi-elliptic. So we have $\chi(\mathcal{O}_{S/\sigma}) = 0$. On the other hand, $q(S/\sigma) = g(B) \leqslant 1$ (Lemma 2.2(iii)), $p_g(S/\sigma) = p_g(S) = 1$, and hence $\chi(\mathcal{O}_{S/\sigma}) \geqslant 1$. This is a contradiction.

Case 2. $\chi(\mathcal{O}_S) = 1$. In this case $g(C) = 1$ and $\bar{\sigma}$ must be a translation of C. This implies that $S \to S/\sigma$ is étale, and hence $1 = \chi(\mathcal{O}_S) = r\chi(\mathcal{O}_{S/\sigma})$, a contradiction.

Case 3. $\chi(\mathcal{O}_S) = 2$. In this case $g(C) = 0$. Then there are exactly two $\bar{\sigma}$-fixed points in C, say t_1, t_2. Let $F_j = f^*t_j$. Applying the topological Lefschetz formula to σ, we have

$$24 = e\big(S^\sigma\big) = e\big(F_1^\sigma\big) + e\big(F_2^\sigma\big). \tag{2.2.1}$$

Let $l(F')$ be the number of irreducible components of F'.

Lemma 2.3. *We have* $e(F_j^\sigma) \leqslant l(F_j) + 1$.

Proof. If F_j is of type ${}_m\mathrm{I}_0$, since the order of σ is a prime larger than three, we have that $\sigma_{|F_j}$ is either identity or a translation. So the lemma is trivially true.

If F' is of type ${}_m\mathrm{I}_b$ with $b > 0$, then $\sigma_{|F_j}$ has k fixed curves and $b - 2k$ isolated fixed points for some k $(0 \leqslant k \leqslant [\frac{b}{2}])$. So $e(F_j^\sigma) = 2k + b - 2k = b = l(F_j)$.

If F' is of type II, III or IV, one checks easily that the lemma is true.

If F' is of type I_b^*, II^*, III^* or IV^*, then there is an irreducible curve $\Gamma < F'$ such that the closure of $F'_{\mathrm{red}} - \Gamma$ has three disjoint connected components and they meet Γ at three different points. By (1.5.1), there are at least three σ-fixed points on Γ. Since Γ is rational, we have that Γ is σ-fixed. Since each σ-fixed curve is contained in the fixed part Z of $|K_S|$ (cf. [Ca1, 1.14]), So $\Gamma < Z$. On the other hand, Since $g(C) = 0$, by (1.1.1), $Z = \sum_{i=1}^k (m_i - 1)D_i$, where $m_1D_1, \dots, m_kD_k$ are all the multiple fibers of f. This implies F_j is of type ${}_m\mathrm{I}_b$, a contradiction. □

By (2.2.1) and Lemma 2.3, we have $l(F_1) + l(F_2) \geqslant 22$. On the other hand, we have

$$2 + \sum_{F'}\big(l(F') - 1\big) \leqslant \operatorname{rank}\mathrm{NS}(S) \leqslant h^{1,1}(S) = 20, \tag{2.3.1}$$

where the sum is taken all singular fibers F' of f. This implies $l(F_1) + l(F_2) \leqslant 20$, a contradiction.

Table 2

No.	$F' = f^*t$	The action of σ on F'
1	${}_m\mathrm{I}_0$	$\sigma_{\vert F'} = \mathrm{id}$
2	${}_m\mathrm{I}_0$	$\sigma_{\vert F'}$ is a translation
3	I_0	$\sigma_{\vert F'}$ has 3 isolated fixed points of weight $\frac{1}{3}(1,2)$
4	I_0	$\sigma_{\vert F'}$ has 3 isolated fixed points of weight $\frac{1}{3}(1,1)$
5	${}_m\mathrm{I}_b$ $(b > 0)$	$\sigma_{\vert F'}$ has b isolated fixed points of weight $\frac{1}{3}(1,2)$
6	${}_m\mathrm{I}_{3n}$ $(n > 0)$	$\sigma_{\vert F'}$ has n fixed curves and n isolated fixed points of weight $\frac{1}{3}(1,1)$
7	III	$\sigma_{\vert F'}$ has 3 isolated fixed points of weight $\frac{1}{3}(1,2)$
8	I^*_{3n}	$\sigma_{\vert F'}$ has $n+1$ fixed curves and $n+4$ isolated fixed points of weight $\frac{1}{3}(1,1)$
9	III*	$\sigma_{\vert F'}$ has 3 fixed curves and 3 isolated fixed points of weight $\frac{1}{3}(1,1)$

Now we assume that $r = 3$. We have a commutative diagram

$$\begin{tikzcd}
S' \arrow[d, "\rho'"] & \tilde{S} \arrow[l, "\tilde{\rho}"'] \arrow[r, "\tilde{\pi}"] \arrow[d] \arrow[dl, "\rho"] & \tilde{T} := \tilde{S}/\tilde{\sigma} \arrow[r, "\eta"] \arrow[d] & T \arrow[dl, "h"] \\
S \arrow[r, "f"'] & C \arrow[r, "\pi"] & B := C/\bar{\sigma} &
\end{tikzcd}$$

where ρ' is the blowup of all isolated fixed points of σ, $\tilde{\rho}$ is the blowup of all isolated fixed points of the lift σ' of σ (for each σ-isolated fixed point $p \in S$ of weight $\frac{1}{3}(1,2)$, there are precisely two isolated σ'-fixed points of weight $\frac{1}{3}(1,1)$ on the exceptional curve $\rho'^{-1}(p)$), $\tilde{\sigma}$ is the lift of σ' on $\tilde{S}$, and η is the blowdown of all -1-curves contained in fibers of $\tilde{S}/\tilde{\sigma} \to B$.

Lemma 2.4. *Let $t \in C$ be a $\bar{\sigma}$-fixed point, and $F' = f^*t$. Then the possible types for F' and the action of σ on F' are given in Table* 2.

Proof. We use repeatedly the following simple observation.

(2.4.1) Let $\Theta < F'$ be a -2-curve. If Θ is not σ-fixed, then among the two σ-fixed points on Θ, either both of them are isolated of weight $\frac{1}{3}(1,2)$, or one is isolated of weight $\frac{1}{3}(1,1)$ and the other is not isolated.

Proof of (2.4.1): Indeed, let $\tilde{\Theta}$ be the strict transform of Θ in $\tilde{S}$, and D the image of $\tilde{\Theta}$ under the quotient map $\tilde{\pi}$. From $\tilde{\Theta} = \tilde{\pi}^*D$, we get $\tilde{\Theta}^2 = 3D^2$. On the other hand, if both of the two σ-fixed points on Θ are isolated of weight $\frac{1}{3}(1,1)$ (resp. not isolated), we have $\tilde{\Theta}^2 = -4$ (resp. -2). This is a contradiction.

We distinguish several cases according to types of F':

(1) F' is of type ${}_m\mathrm{I}_0$. We have that $\sigma_{\vert F'_{\mathrm{red}}}$ is either identity, a translation or an automorphism with three fixed points. In the first or second case, we have (F', σ) is as in (1) or (2) of Table 2. In the third case, if $m > 1$, then $h^*\pi(t)$ is a multiple fiber with $h^*\pi(t)_{\mathrm{red}}$ being simply connected; this is absurd. So we have that $m = 1$ and (F', σ) is as in (3) or (4) of Table 2.

(2) F' is of type ${}_m\mathrm{I}_b$ $(b > 0)$. If $b = 1$, by the argument as in the proof of (2.4.1), the action of σ at the singular point of F'_{red} is of weight $\frac{1}{3}(1,2)$, and hence (F', σ) is as in (5) of Table 2 with $b = 1$.

Now we assume that $b > 1$. Then F'_{red} is a cycle of smooth rational curves. We label irreducible components $\Theta_0, \Theta_1, \dots, \Theta_{b-1}$ of F' with subscripts mod b so that $\Theta_i\Theta_{i+1} = 1$. Let $p_i = \Theta_i \cap \Theta_{i+1}$. By (2.4.1), if Θ_i is σ-fixed for some i, then the action of σ at p_{i-2} and p_{i+1} is of weight $\frac{1}{3}(1,1)$; if the action of σ at p_i is of weight $\frac{1}{3}(1,1)$ for some i, then Θ_{i-2} and Θ_{i+1} are σ-fixed curves. So (F', σ) is as in (6) or (5) of Table 2 depending on whether there are σ-fixed curves contained in F' or not.

(3) F' is of type II. Let $p \in F'$ be the singular point. Then p is isolated σ-fixed. Let Θ' be the strict transform of F' in S', $E' = \rho'^{-1}(p)$, and $p' = \Theta' \cap E'$. If the action of σ at p is of weight $\frac{1}{3}(1,1)$, then

E' is σ'-fixed. Since E' and Θ' have the same tangent space at p', we have that $\sigma'_{|\Theta'} = \mathrm{id}$, and hence the restriction of σ' on the connected curve $\Theta' + E'$ is trivial, which is absurd.

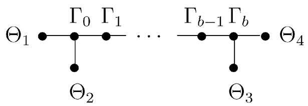
So we may assume that the action of σ at p is of weight $\frac{1}{3}(1,2)$. Then the action of σ' at p' is of weight $\frac{1}{3}(1,1)$. Let $\tilde{\Theta}'$ (resp. $\tilde{E}'$) be the strict transform of Θ' (resp. E') in $\tilde{S}$ and $\tilde{E} = \tilde{\rho}^{-1}(p')$. Then $\tilde{E}$, $\tilde{E}'$ and $\tilde{\Theta}'$ meet transversally at a point, say $\tilde{p}$. This implies that there are at least three 1-dimensional $\tilde{\sigma}$-invariant subspaces in $T_{\tilde{p}}\tilde{S}$, and hence the action of $\tilde{\sigma}$ at $\tilde{p}$ is of weight $\frac{1}{3}(1,1)$. On the other hand, Since $\tilde{E}$ is $\tilde{\sigma}$-fixed, the action of $\tilde{\sigma}$ at $\tilde{p}$ is not isolated. This is a contradiction.

(4) F' is of type III. We have that the singular point p of F' is isolated σ-fixed. Indeed, since the two irreducible components of F' have the same tangent space at p, if one of them is σ-fixed then the other and hence F' is also σ-fixed, which is absurd.

If the action of σ at p is of weight $\frac{1}{3}(1,1)$, we get a contradiction as in case (3) with F' instead of $\Theta' + E'$. So the action of σ at p and hence at each σ-fixed point is of weight $\frac{1}{3}(1,2)$, and (F', σ) is as in (6) of Table 2.

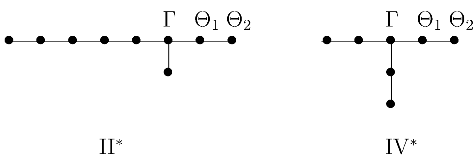
(5) F' is of type IV. Write $F' = \Theta_1 + \Theta_2 + \Theta_3$. Let p be the singular point of F'. If one of the three irreducible components of F' is σ-fixed, we get a contradiction as in case (3) with F' instead of $\tilde{\Theta}' + \tilde{E}' + \tilde{E}$. So we may assume that there are no σ-fixed curves contained in F'. Since there are at least three 1-dimensional σ-invariant subspaces $T_p\Theta_i$ $(i = 1,2,3)$ in T_pS, the action of σ at p is of weight $\frac{1}{3}(1,1)$. This is contrary to (2.4.1).

(6) F' is of type I_b^* *($b \geqslant 0$)*. The dual graph of F' is as below:



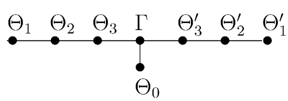
By (1.5.1), there are at least three σ-fixed points on Γ_0. Since Γ_0 is rational, we have that Γ_0 (and Γ_b) is σ-fixed. By (2.4.1), we have that $p_1,\ p_4,\ p_7,\ \ldots$ (where $p_i := \Gamma_i \cap \Gamma_{i+1}$) are isolated points of weight $\frac{1}{3}(1,1)$, $\Gamma_3,\ \Gamma_6,\ \Gamma_9,\ \ldots$ are σ-fixed, $b \equiv 0 \bmod 3$, and there is precisely one isolated σ-fixed point of weight $\frac{1}{3}(1,1)$ on Θ_i for $i = 1,\ldots,4$. So (F', σ) is as in (7) of Table 2.

(7) F' is of type II* *or* IV*. The dual graph of F' is as below:



We have that Γ is σ-fixed. By (2.4.1), the action of σ at $p := \Theta_1 \cap \Theta_2$ is of weight $\frac{1}{3}(1,1)$. Applying (2.4.1) to Θ_2, we get a contradiction.

(8) F' is of type III*. The dual graph of F' is as below:



We have that Γ is σ-fixed. By (2.4.1), the action of σ at $p := \Theta_2 \cap \Theta_3$ $p' := \Theta'_2 \cap \Theta'_3$ is of weight $\frac{1}{3}(1,1)$, Θ_1 and Θ'_1 are σ-fixed, and there is precisely one isolated σ-fixed point of weight $\frac{1}{3}(1,1)$ on Θ_0. So (F', σ) is as in (9) of Table 2. $\square$

By Lemma 2.4, we have Table 3.

Table 3

No	1	2	3	4	5	6	7	8	9
Type of F'	${}_m\mathrm{I}_0$	${}_m\mathrm{I}_0$	I_0	I_0	${}_m\mathrm{I}_b$	${}_m\mathrm{I}_{3n}$	III	I_{3n}^*	III*
$e(F')$	0	0	0	0	b	$3n$	3	$3n+6$	9
$e(F'^{\sigma})$	0	0	3	3	b	$3n$	3	$3n+6$	9
$l(F')$	1	1	1	1	b	$3n$	2	$3n+5$	8

The proof of (ii) of Proposition 2.1 continues. Let $\mathcal{I}$ be the set of σ-invariant fibers of f. For $n \geqslant 1$, let x_b (resp. y_n, z_n) be the number of fibers of type ${}_m\mathrm{I}_b$ (resp. ${}_m\mathrm{I}_{3n}$, ${}_m\mathrm{I}_{3n}^*$) in $\mathcal{I}$ as in No. 5 (resp. No. 6, No. 8) of Table 2, and for $i = 1, 2, 3, 4, 7, 9$, let u_i be the number of fibers in $\mathcal{I}$ as in No. i of Table 2. Then we have

$$\sum_b x_b + \sum_n y_n + \sum_n z_n + \sum_i u_i = g(C) - 3g(B) + 2. \tag{2.4.1}$$

Let w_1 (resp. w_2) be the number of fibers of f of type ${}_m\mathrm{I}_1$ (resp. II) that are not in $\mathcal{I}$. Then we have that both w_1 and w_2 divide 3.

By (iv) of Lemma 2.2, we have that $\mathrm{Trace}(\sigma|H^1(S,\mathbb{C})) = 3g(B) - g(C)$. Since $H^1(S,\mathbb{C})$ and $H^3(S,\mathbb{C})$ are dual σ-spaces, we have $\mathrm{Trace}(\sigma|H^3(S,\mathbb{C})) = \mathrm{Trace}(\sigma|H^1(S,\mathbb{C}))$. Applying the topological Lefschetz formula to σ (cf. e.g. [Ue]), we have

$$e\big(S^{\sigma}\big) = 2 - 2\big(3g(B) - g(C)\big) + b_2(S) = e(S) - 6\big(g(B) - g(C)\big).$$

By Table 3, we have

$$\begin{aligned} 3u_3 + 3u_4 &= \sum_{F' \in \mathcal{I}} \big(e\big(F'^{\sigma}\big) - e(F')\big) \\ &= e\big(S^{\sigma}\big) - \big(e(S) - w_1 - 2w_2\big) \\ &= 6\big(g(C) - g(B)\big) + w_1 + 2w_2. \end{aligned} \tag{2.4.2}$$

Note that $\mathrm{def}_p = -\frac{2}{3}$ (resp. $\frac{2}{3}$) if the action of σ at p is of weight $\frac{1}{3}(1,1)$ (resp. $\frac{1}{3}(1,2)$). Applying the equivariant signature formula to σ (1.6.1), we have

$$\begin{aligned} -16\chi(\mathcal{O}_S) = \frac{-16}{3}\bigg(\sum_n n y_n + \sum_n (n+1) z_n + 3u_9\bigg) + \frac{2}{3}\bigg(3u_3 + \sum_b b x_b + 3u_7\bigg) \\ + \bigg(-\frac{2}{3}\bigg)\bigg(3u_4 + \sum_n n y_n + \sum_n (n+4) z_n + 3u_9\bigg). \end{aligned}$$

So we obtain

$$\sum_n 3n y_n + \sum_n (3n+4) z_n + 9u_9 + u_4 = 8\chi(\mathcal{O}_S) + \frac{\sum_b b x_b}{3} + u_3 + u_7. \tag{2.4.3}$$

Now we distinguish three cases according to $\chi(\mathcal{O}_S)$:

Case 1. $\chi(\mathcal{O}_S) = 0$. Since f has no fibers of positive Euler topological characteristic in this case, Σ consists of fibers of Nos. 1–4 of Table 2. By (2.4.3), we have $u_3 = u_4$. Combining this with (2.4.2), we have $u_4 = g(C) - g(B) \geqslant 2$, here the last inequality holds by (iii) of Lemma 2.2. Note that for an isolated σ-fixed point $p \in S$, if the action of σ at p is not of weight $\frac{1}{3}(1,2)$, then p is in the base locus of $|K_S|$ (see e.g. [Pet1, Lemma 2]). We have that each fiber of No. 4 of Table 2 is contained in the fixed part of $|K_S|$, and so the degree of the moving part of $|\mathcal{L}\otimes\omega_C|$ is at most $2g(C)-2-u_4$, where $\mathcal{L}$ is as in (1.1). Since $p_g(S)\geqslant 2$ by assumption, by the proof of Lemma 1.2, we get

$$|Q|\leqslant\frac{2g(C)-2-u_4}{g(C)-2}\leqslant\frac{2g(C)-4}{g(C)-2}=2.$$

Case 2. $\chi(\mathcal{O}_S)=1$. By (2.4.1), we have that $u_3+u_4\leqslant g(C)-3g(B)+2$. If the equality holds, then Σ consists of fibers of Nos. 3–4 of Table 2. By (2.4.3), we have that $u_4=8+u_3$, a contradiction. Now we may assume that $u_3+u_4<g(C)-3g(B)+2$. Combining this with (2.4.2), (note that $g(C)=p_g(S)\geqslant 1$ by assumation,) we have that $g(C)=1$, $g(B)=0$, $u_3+u_4=2$, and $w_1=w_2=0$. Combining this with (2.4.1), we have that there is exactly one fiber of f, say F_1, with positive Euler topological characteristic. So $e(F_1)=e(S)=12$. By Table 3, F_1 is either of No. 5 of Table 2 with $b=12$, or of No. 6 with $n=4$, or of No. 8 with $n=2$. By (2.4.3), we get a contradiction.

Case 3. $\chi(\mathcal{O}_S)=2$. By Lemma 1.2(ii), we may assume that $g(C)=0$. Then there are exactly two $\bar\sigma$-fixed points in C, say t_1, t_2. Let $F_j=f^*t_j$. We have (cf. (2.2.1))

$$e\big(F_1^\sigma\big)+e\big(F_2^\sigma\big)=24. \tag{2.4.4}$$

By Table 3, we have $e(F_j^\sigma)\leqslant l(F_j)+1$ except that (F',σ) is as in (3) or (4) of Table 2. If neither F_1 nor F_2 are as in (3) or (4) of Table 2, we have $l(F_1)+l(F_2)\geqslant 21$; if one of them (not both), say F_1, is as in (3) or (4) of Table 2, by (2.4.4) we have $e(F_2^\sigma)=21$, and so $l(F_2)=20$ by Table 3; if both of them are as in (3) or (4) of Table 2, we have $e(F_1^\sigma)+e(F_2^\sigma)=6$, which is contrary to (2.4.4). So we always have $l(F_1)+l(F_2)\geqslant 21$. By (2.3.1), we get a contradiction.

This completes the proof of (ii) of Proposition 2.1. □

3. $\chi(\mathcal{O}_S)\leqslant 2$

Theorem 3.1. *Let $f:S\to C$, G be as in* (1.1)*.*

(i) *If $\chi(\mathcal{O}_S)=2$, then G is trivial.*
(ii) *If $\chi(\mathcal{O}_S)=1$ and $p_g(S)>0$, then $|G|\leqslant 2$. Moreover, if $|G|=2$, the generator σ of G induces a* 0*-involution $\bar\sigma$ of C such that $\bar\sigma\circ f=f\circ\sigma$, and there are four possible configurations of singular fibers of f:*

$${}_m\mathrm{I}_8+4{}_m\mathrm{I}_1,\quad \mathrm{IV}^*+\mathrm{IV},\quad \mathrm{IV}^*+4{}_m\mathrm{I}_1,\quad \mathrm{IV}^*+2\mathrm{II}.$$

Here the subscript m may be different for different fibers, and multiple fibers of f of type ${}_m\mathrm{I}_0$ over $C\setminus\{\bar\sigma$-fixed points$\}$ are not listed in the configurations.
(iii) *Assume that $\chi(\mathcal{O}_S)=0$, $p_g(S)\geqslant 3$, and that S has no non-trivial holomorphic vector fields. Let H' be as in* (ii) *of Proposition* 1.7*. Then either $G=H'$ or H' is a normal subgroup of G of index* 2*. In the latter case, for each $\sigma\in G\setminus H'$, σ induces a ι-involution* ($\iota=0$ *or* 1) *$\bar\sigma$ of C such that $\bar\sigma\circ f=f\circ\sigma$, and G is isomorphic to one of the following four groups*: $\mathbb{Z}_2$, $\mathbb{Z}_2^{\oplus 2}$, Sym(3), $\mathbb{Z}_2^{\oplus 3}$.

Here an involution σ of a curve C is called ι-involution ($\iota\in\mathbb{Z}$, $\iota\geqslant 0$), if $g(C/\sigma)=\iota$. When $g(C)\geqslant 2$, "0-involution" is just "the hyperelliptic involution" and "1-involution" is just "bi-elliptic involution".

Table 4

No.	$F' = f^*t$	The action of σ on F'	$H' = h^*\pi(t)$
1	I_0	$\sigma_{\mid F'}$ has 4 isolated fixed points	I_0^*
2	${}_m\mathrm{I}_0$	$\sigma_{\mid F'} = \mathrm{id}$	${}_m\mathrm{I}_0$
3	${}_m\mathrm{I}_0$	$\sigma_{\mid F'}$ is a translation	${}_{2m}\mathrm{I}_0$
4	${}_m\mathrm{I}_{2n}$ $(n \geqslant 1)$	$\sigma_{\mid F'}$ has n fixed curves	${}_m\mathrm{I}_n$
5	IV	$\sigma_{\mid F'}$ has four isolated fixed points	IV*
6	IV*	$\sigma_{\mid F'}$ has four fixed curves	IV

Proof of (i)–(ii) of Theorem 3.1. Assume that Q is not trivial. Let $\sigma \in G$ such that its image $\bar\sigma \in Q$ is an element of prime order. By Proposition 2.1, σ is an involution. We have a commutative diagram

$$\begin{tikzcd}
S \arrow[dr, "f"'] & \tilde{S} \arrow[l, "\rho"'] \arrow[r, "\tilde{\pi}"'] \arrow[d] & \tilde{T} := \tilde{S}/\tilde{\sigma} \arrow[r, "\eta"'] \arrow[d] & T \arrow[dl, "h"] \\
& Cs \arrow[r, "\pi"] & B := C/\bar{\sigma} &
\end{tikzcd}$$

where ρ is the blowup of all isolated fixed points of σ, $\tilde\sigma$ the induced involution on $\tilde S$, and η is the blowdown of all -1-curves contained in fibers of $\tilde S/\tilde\sigma \to B$. Then $h : T \to B$ is a relatively minimal elliptic fibration.

For each $t \in C$, f^*t is σ-invariant if and only if t is $\bar\sigma$-fixed.

Lemma 3.2. *Let $t \in C$ be a $\bar\sigma$-fixed point, and $F' = f^*t$. Then the possible types for F', the action of σ on F' and the types for $H' := h^*\pi(t)$ are given in Table 4.*

Proof. We distinguish five cases according to types of F':

(1) F' is of type ${}_m\mathrm{I}_0$. By the similar argument as in case (1) of Lemma 2.4, we have that (F', σ) is as in (1)–(3) of Table 4.

(2) F' is of type II *or* III. Let p be the singular point of F'. Note that if p is isolated σ-fixed, then $\mathrm{mult}_p f^*(f(p))$ is odd [Ca2, II, Lemma 2.4]. Since $\mathrm{mult}_p F'$ is even, we will get a contradiction by showing that p is isolated σ-fixed. This is trivially true if F' is of type II and true by the argument as in case 4 of Lemma 2.4 if F' is of type III.

(3) F' is of type ${}_m\mathrm{I}_b$. If b is odd, by (1.5.1), we see easily that among the nodes of F'_{red}, there exists an isolated σ-fixed point. We exclude this case as in case (2). Hence we may assume that b is even and that σ has no isolated σ-fixed points on F'. So we have (F', σ) is as in (4) of Table 4.

(4) F' is of type IV. If F' contains no σ-fixed curves, then there are exactly four isolated σ-fixed points on F', and so (F', σ) is as in (5) of Table 4; if F' contains a σ-fixed curve, then the other two irreducible curves of F' are not σ-fixed and each of them contains exactly one isolated σ-fixed point. This contrary to the following simple observation:

(3.2.1) Let $\Theta < F'$ be a -2-curve. If Θ is not σ-fixed, then there are exactly two isolated σ-fixed points on Θ.

Proof of (3.2.1). Indeed, let $\tilde\Theta$ be the strict transform of Θ in $\tilde S$, and D the image of $\tilde\Theta$ under the quotient map $\tilde\pi$. If there is precisely one isolated σ-fixed point on Θ, $\tilde\Theta^2 = -3$. On the other hand, From $\tilde\Theta = \tilde\pi^*D$, we get $\tilde\Theta^2 = 2D^2$. This is a contradiction.

(5) F' is of type I_b^* $(b \geqslant 0)$, II*, III* *or* IV*. There is an irreducible curve $\Gamma < F'$ such that the closure of $F'_{\mathrm{red}} - \Gamma$ has three disjoint connected components and they meet Γ at three different points. By the argument as in case (6) of Lemma 2.4, we have that Γ is σ-fixed. If F' is not of type IV*, then there is an irreducible curve $\Theta < F'$ such that $\Theta\Gamma = 1$ and Θ does not meet any other curve contained in F'. This implies Θ is not σ-fixed and there is exactly one isolated σ-fixed point on Θ,

which is contrary to (3.2.1). If F' is of type IV*, by (3.2.1), we have that the three irreducible curves that do not meet Γ are also σ-fixed. So (F', σ) is as in (6) of Table 4. $\square$

The proof of Theorem 3.1 continues. Let $\mathcal{I}$ be the set of σ-invariant fibers of f. For $n \geqslant 1$, let x_n be the number of fibers of type ${}_m\mathrm{I}_{2n}$ in $\mathcal{I}$ as in No. 4 of Table 4, and for $i = 1, 2, 3, 5, 6$, let y_i be the number of fibers in $\mathcal{I}$ as in No. i of Table 4. Then we have

$$\sum_n x_n + y_1 + y_2 + y_3 + y_5 + y_6 = 2g(C) - 4g(B) + 2. \tag{3.2.1}$$

For each singular fiber F' of f, if F' is not σ-invariant, then F'_{red} must be irreducible and it is of type ${}_m\mathrm{I}_1$, II, or ${}_m\mathrm{I}_0$. On the other hand, by Lemma 3.2, we have that any fiber of type ${}_m\mathrm{I}_1$ or II is not σ-invariant. So fibers of type ${}_m\mathrm{I}_1$ or II appear in pairs. Let z_1 (resp. z_2) be the number of fibers of f of type ${}_m\mathrm{I}_1$ (resp. II). Then z_1 and z_2 are even. By Lemma 3.2, we have

$$12\chi(\mathcal{O}_S) = e(S) = \sum_{F' \text{ of } f} e(F') = \sum_n 2nx_n + 4y_5 + 8y_6 + z_1 + 2z_2, \tag{3.2.2}$$

$$e(T) = \sum_{H' \text{ of } h} e(H') = \sum_n nx_n + 6y_1 + 8y_5 + 4y_6 + \frac{z_1}{2} + z_2. \tag{3.2.3}$$

By (3.2.2) and (3.2.3), we have

$$y_1 + y_5 = \frac{e(T) - 6\chi(\mathcal{O}_S)}{6} = \chi(\mathcal{O}_S) + 2\big(g(C) - g(B)\big), \tag{3.2.4}$$

where the last equality holds since $e(T) = 12\chi(\mathcal{O}_T) = 12(\chi(\mathcal{O}_S) + g(C) - g(B))$ by (i)–(ii) of Lemma 2.2. Combining this with (3.2.1), we have

$$\sum_n x_n + y_2 + y_3 + y_6 = 2 - 2g(B) - \chi(\mathcal{O}_S). \tag{3.2.5}$$

Let $\Theta_1, \dots, \Theta_m$ ($m \geqslant 0$) be the σ-fixed curves. Since σ induces non-trivial action on C, Θ_i ($0 \leqslant i \leqslant m$) are contained in singular fibers of f. By Lemma 2.1, we have that Θ_i is either a -2-curve or an elliptic curve, and

$$\sum_i e(\Theta_i) = \sum_n 2nx_n + 8y_6. \tag{3.2.6}$$

We may apply the topological and holomorphic Lefschetz formula to σ (cf. [AS, p. 566]):

$$\begin{aligned} &e(S) + 8\big(q(S) - \dim H^0\big(S, \Omega_S^1\big)^\sigma\big) - 2\big(h^2(S, \mathbb{Q}) - \dim H^2(S, \mathbb{Q})^\sigma\big) \\ &\quad = e\big(S^\sigma\big) = k + \sum_{i=1}^m e(\Theta_i) \\ &\chi(\mathcal{O}_S) + 2\big(q(S) - \dim H^0\big(S, \Omega_S^1\big)^\sigma\big) = \frac{k - \sum_{i=1}^m K_S D_i}{4}, \end{aligned} \tag{3.2.7}$$

Table 5

No.	Configurations of singular fibers of h
1	${}_m\mathrm{I}_4 + \mathrm{IV}^* + 2g\mathrm{I}_0^*$
2	${}_m\mathrm{I}_4 + \mathrm{II} + (2g+1)\mathrm{I}_0^*$
3	${}_m\mathrm{I}_4 + 2{}_m\mathrm{I}_1 + (2g+1)\mathrm{I}_0^*$
4	$\mathrm{IV} + \mathrm{IV}^* + 2g\mathrm{I}_0^*$
5	$\mathrm{IV} + 2{}_m\mathrm{I}_1 + (2g+1)\mathrm{I}_0^*$
6	$\mathrm{IV} + \mathrm{II} + (2g+1)\mathrm{I}_0^*$

where k is the number of isolated σ-fixed points. We get

$$K_S^2 = 8\chi(\mathcal{O}_S) - \sum_{i=1}^{m} e(\Theta_i). \tag{3.2.8}$$

By (3.2.6) and (3.2.8), we have

$$\sum_n n x_n + 4y_6 = 4\chi(\mathcal{O}_S). \tag{3.2.9}$$

If $\chi(\mathcal{O}_S) = 2$, by (3.2.5) and (3.2.9), we get a contradiction. This finishes the proof of (i) of Theorem 3.1.

Now we assume $\chi(\mathcal{O}_S) = 1$. Then We have

$$\begin{gathered} x_4 + y_6 = 1, \qquad y_2 = y_3 = 0, \qquad x_i = 0 \quad \text{for } i \neq 4 \quad \bigl(\text{by (3.2.5) and (3.2.9)}\bigr), \\ y_1 + y_5 = 1 + 2g(C) \quad \bigl(\text{by (3.2.4)}\bigr), \\ 4y_5 + z_1 + 2z_2 = 4 \quad \bigl(\text{by (3.2.2)}\bigr). \end{gathered}$$

So there are six possible configurations of singular fibers of h, where $g = g(C)$:

Here the subscript m may be different for different fibers, multiple fibers of h of type ${}_m\mathrm{I}_0$ corresponding to that of $f_{|C^o} : f^{-1}(C^o) \to C^o$ ($C^o = C \setminus \{\bar{\sigma}\text{-fixed points}\}$) are not listed in the configurations, and for example, ${}_m\mathrm{I}_4 + 2{}_m\mathrm{I}_1 + (2g+1)\mathrm{I}_0^*$ means that h has one fiber of type ${}_m\mathrm{I}_4$, two fibers of type ${}_m\mathrm{I}_1$ and $2g+1$ fibers of type I_0^*.

Case 1. No. 1 of Table 5 does not occur.

Let h be as in No. 1 of Table 5, and H a general fiber of h. The Picard–Lefschetz transformation around the singular of type ${}_m\mathrm{I}_4$ can be given with respect to suitable basis of $H_1(H,\mathbb{Z})$ by $\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$. Under such basis, the Picard–Lefschetz transformation around the singular fiber of type IV^* is $N\left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)N^{-1}$ for some $N = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, and that of type I_0^* is $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{2g} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} N \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} N^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

By an elementary calculation, we have $c = d = 0$, which is absurd. So No. 1 of Table 5 does not occur.

Case 2. No. 2 of Table 5 does not occur.

If h is as in No. 2 of Table 5, by the monodromy argument as above, we have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{2g+1} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} N \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} N^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some $N \in SL(2,\mathbb{Z})$. We get a contradiction as in Case 1. So we obtain the possible configurations of singular fibers of f as in (ii) of Theorem 3.1.

To finish the proof of (ii) of Theorem 3.1, we show that $|Q| = 2$. By Proposition 2.1, we have that $|Q| = 2^n$ for some natural number n. If $p_g(S) \geqslant 2$, by Lemma 1.2(ii), we have that $n = 1$. So we may assume that $p_g(S) = 1$. Then C is an elliptic curve, and so $Q = T \rtimes A \subseteq \operatorname{Aut} C$, where T is a group of translations and $A \subseteq \operatorname{Aut} C$ is a subgroup preserving the group structure.

We claim that T is trivial. Otherwise, there is an automorphism $\alpha \in G$ such that its image $\bar{\alpha} \in Q$ being a non-trivial translation of C. Then $S \to S/\alpha$ is étale, and hence $1 = \chi(\mathcal{O}_S) = o(\alpha)\chi(\mathcal{O}_{S/\alpha})$, a contradiction.

So $G \simeq Q = A$ is a cyclic group of order 2 or 4. Now suppose that G is a cyclic group of order 4. Let α be a generator of G, and $\bar{\alpha} \in Q$ its image. There are exactly two $\bar{\alpha}$-fixed points in C, say t_1, t_2. Let $F_j = f^*t_j$. Clearly F_j is α-invariant, and $S^\alpha = F_1^\alpha \cup F_2^\alpha$.

If F_j is of type IV*, note that the action of α^2 on F_j is as No. 6 of Table 4, we have that there are $1 + k$ α-fixed curves and $2(3-k)$ isolated α-fixed points for some k $(0 \leqslant k \leqslant 3)$, and hence $e(F_j^\alpha) = 2(1+k) + 2(3-k) = 8$. Similarly, if F_j is of type ${}_m\mathrm{I}_8$ (resp. I_0, IV), we have $e(F_j^\alpha) = 8$ (resp. 2, 4).

By the possible configurations of α^2-invariant fibers of f as in (ii) of Theorem 3.1, we have that the possible configurations of $F_1 + F_2$ are ${}_m\mathrm{I}_8 + \mathrm{I}_0$, IV* + IV, or IV* + I_0. So $e(F_1^\alpha) + e(F_2^\alpha) = 10$, or 12. On the other hand, by the same argument in the proof of Lemma 2.2(iv), we have that $\operatorname{Trace}(\sigma | H^1(S,\mathbb{C})) = 0$. Applying the topological Lefschetz formula to α, we have

$$e\big(F_1^\alpha\big) + e\big(F_2^\alpha\big) = e\big(S^\sigma\big) = 2 + b_2(S) = e(S) + 4g(C) = 16.$$

This is a contradiction.

Proof of (iii) of Theorem 3.1. By (ii) of Proposition 1.7, we have that H' is a normal subgroup of G and $Q \simeq G/H'$. By Proposition 2.1, we have that $|Q| = 2^n$ for some natural number n. Since $p_g(S) \geqslant 3$ by assumption, we have that $n = 1$ by Lemma 1.2(ii).

By (i) of Proposition 2.1, if H' is isomorphic to a trivial group (resp. $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2^{\oplus 2}$), then G is isomorphic to $\mathbb{Z}_2$ (resp. $\mathbb{Z}_2^{\oplus 2}$, Sym(3), $\mathbb{Z}_2^{\oplus 3}$). The remaining part is clear by (iii) of Lemma 2.2. □

4. Examples

4.1. Proposition-Construction

Let $g \geqslant 0$ be an integer, and let $h: T \to B := \mathbb{P}^1$ be an elliptic fiber surface with the configuration of singular fibers given as in one of Nos. 3–6 of Table 5. Let $\pi: C \to B$ be the double cover branch along points which are the image of all fibers H' of h with $e(H') > 1$.

Let S' be the desingularization of $T \times_B C$, and $f': S' \to C$ the induced fibration. Let $f: S \to C$ be the relatively minimal fiberation of f'. Let σ the involution corresponding to the double cover $T \times_B C \to T$. Then $\chi(\mathcal{O}_S) = 1$, $p_g(S) = q(S) = g$ and σ acts trivially on $H^2(S,\mathbb{Q})$.

Proof. It follows by the converse process of the proof of (ii) of Theorem 3.1. Indeed, for each ramification point $t \in C$, if $h^*(\pi(t))$ is of type I_0^* (resp. ${}_m\mathrm{I}_4$, IV*, IV), the type for f^*t and the action of σ on f^*t is as in No. 1 (resp. 4, 5, 6) of Table 4. Consequently, we may calculate $e(S^\sigma)$. From $e(S) = \sum_{F'} e(F')$, where the sum is taken over all singular fiber of f, we have $e(S) = 12$, and hence $\chi(\mathcal{O}_S) = 1$. Since $\deg(R^1 f_*\mathcal{O}_S)^\vee = \chi(\mathcal{O}_S) > 0$, We have $q(S) - g(C) = h^0(R^1 f_*\mathcal{O}_S) = 0$. So f induces an isomorphism $f^*: H^1(C,\mathbb{C}) \to H^1(S,\mathbb{C})$, which is compatible with the induced action of σ, and hence $\sigma^* = -\mathrm{id}$ on $H^1(S,\mathbb{C})$. Now applying the topological Lefschetz formula to σ (cf. (3.2.7)), we get $\dim H^2(S,\mathbb{Q})^\sigma = h^2(S,\mathbb{Q})$. $\square$

By (4.1), we reduces the existence of (S,G) in (ii) of Theorem 3.1 to that of fiber surfaces $h: T \to B$ with type of singular fibers given as in one of Nos. 3–6 of Table 5.

Example 4.2 *(No. 3 of Table 5).* Let $P = \mathbb{P}^1 \times \mathbb{P}^1$, $\psi_1 : P \to \mathbb{P}^1$ be the projection to the first factor, and let x, y be inhomogeneous parameters of the factors of P.

Let C_1, C_2 be curves in P defined by equations as follows:

$$C_1 : x = y + xy, \qquad C_2 : x = y + 1.$$

Then C_1 and C_2 intersect at points $(a, a-1)$, $(1-a, -a)$, where $a = \frac{1+\sqrt{5}}{2}$.

Let D_i, R_i $(i = 1,2,3)$ be curves in P defined by equations as follows:

$$D_1 : (a-1)x = ay, \qquad D_2 : -ax = (1-a)y, \qquad D_3 : x = \sqrt{-1},$$
$$R_1 : \frac{\sqrt{-1}+1}{2}x = \sqrt{-1}y, \qquad R_2 : (\sqrt{-1}-1)x = \sqrt{-1}y, \qquad R_3 : xy = 1.$$

Let $D = D_1 + D_2 + D_3$ and $R = R_1 + R_2 + R_3$. Then the divisor D (resp. R) passes through points $(0,0)$, (∞,∞) $(\sqrt{-1}, \frac{\sqrt{-1}+1}{2})$, $(\sqrt{-1}, \sqrt{-1}-1)$, $(a, a-1)$ and $(1-a, -a)$, and the multiplicity of D (resp. R) at $(0,0)$, (∞,∞) is two. Let Λ be the pencil in $|(3,2)|$ generated by D and R, and let C_3 be a general divisor of Λ. By Bertini's theorem, C_3 is irreducible and has exactly two singularities (nodes at $(0,0)$, (∞,∞)). Let $\nu : \tilde{C_3} \to C_3$ be the normalization of C_3. Then $\tilde{C_3}$ is rational. Let $b_1, b_2 \in \mathbb{P}^1$ be the branch points of the double cover $\nu \circ \psi_{1|C_3} : \tilde{C_3} \to \mathbb{P}^1$.

Let $g \geqslant 2$ be an integer. Let $t_1, \dots, t_{2g-3} \in \mathbb{P}^1 \setminus \{b_1, b_2, 0, \infty, \sqrt{-1}, a, 1-a\}$ be $2g-3$ different points, and let $F_i = t_i \times \mathbb{P}^1$. Let $B = C_1 + C_2 + C_3 + \sum_{i=1}^{2g-3} F_i$. Then $B \in |(2g+2, 4)|$ is an even divisor. Let $\pi : T' \to P$ be the double cover branched along B. Then T' has only canonical singularities. Let $\epsilon : T \to T'$ be the minimal desingularization. Now $h := \psi_1 \circ \pi \circ \epsilon : T \to B$ is an elliptic fibration. By a simple calculation, we have that h has exactly $2g+4$ singular fibers: $2g+1$ fibers of type I_0^* over points $0, \infty, a, 1-a, t_1, \dots, t_{2g-3}$, two fibers of type I_1 over points b_1, b_2, and one fiber of type I_4 over point $\sqrt{-1}$. Hence h is as in No. 3 of Table 5.

Remark 4.3. I do not know if there exist examples of Nos. 4–6 of Table 5 with g arbitrary large; there are such configurations of singular fibers of h with $g = 0$ (cf. [Mi,Per]).

In the end of this section, we construct explicitly pairs (S,G) in (iii) of Theorem 3.1 with $|G| \geqslant 6$; the remaining cases are constructed similarly and are therefore omitted.

Example 4.4 *($\chi(\mathcal{O}_S) = 0$ and $G \simeq \mathrm{Sym}(3)$).* Let $\tilde{B} \to B$ be a cyclic cover of degree 3 of elliptic curves, and $C \to B$ a double cover with $g(C) \geqslant 2$. Let $\tilde{C} = C \times_B \tilde{B}$, and α (resp. β) the automorphism of $\tilde{C}$ corresponding to the cover $\tilde{C} \to C$ (resp. $\tilde{C} \to \tilde{B}$).

Let $F = \mathbb{C}/\mathbb{Z} + \rho\mathbb{Z}$, and $a = \frac{1-\rho}{3}$, where $\rho = e^{\frac{2\pi\sqrt{-1}}{3}}$. Let ϱ be the automorphism of F given by $x \mapsto \rho x$.

Let $S = (\tilde{C} \times F)/\langle \alpha \times \varrho \rangle$. Then the group G generated by $\overline{\beta \times -1_F}$, $\overline{\mathrm{id}_{\tilde{C}} \times t_a}$ is isomorphic to Sym(3) and acts trivially on $H^2(S,\mathbb{Q})$, where t_a is the translation by a and $\bar{\eta}$ is the automorphism of S induced by $\eta \in \mathrm{Aut}(\tilde{C} \times F)$. Indeed, we have

$$\begin{aligned} H^2(S) = H^2(\tilde{C} \times F)^{\alpha \times \varrho} &= H^0(\tilde{C}) \otimes H^2(F) \oplus H^2(\tilde{C}) \otimes H^0(F) \\ &\oplus H^1(\tilde{C})_\alpha^\rho \otimes H^1(F)_\varrho^{\bar{\rho}} \oplus H^1(\tilde{C})_\alpha^{\bar{\rho}} \otimes H^1(F)_\varrho^\rho, \end{aligned} \tag{4.4.1}$$

where $H^i(\dagger) = H^i(\dagger, \mathbb{C})$ and $H^i(\dagger)_\alpha^\zeta$ is the eigensubspace of α-space $H^i(\dagger)$ with eigenvalue ζ.

Clearly $\overline{\mathrm{id}_{\tilde{C}} \times t_a}$ induces trivial action on the right side of (4.4.1). By the construction of $\tilde{C}$, we see easily that both $H^1(\tilde{C})^{\rho}_{\alpha}$ and $H^1(\tilde{C})^{\bar{\rho}}_{\alpha}$ are contained in $H^1(\tilde{C})^{-1}_{\beta}$. So $\overline{\beta \times -1_F}$ induces trivial action on the right side of (4.4.1).

Example 4.5 *($\chi(\mathcal{O}_S) = 0$ and $G \simeq \mathbb{Z}_2^{\oplus 3}$).* Let $\tilde{B} = \mathbb{P}^1$ and $\gamma_{\tilde{B}}$ an involution of $\tilde{B}$. Let $\pi : C \to B := \tilde{B}/\langle \gamma_{\tilde{B}} \rangle$ be a double cover with $g(C) \geqslant 2$, such that the branch points of $\tilde{B} \to \tilde{B}/\langle \gamma_{\tilde{B}} \rangle = B$ are contained in that of π. Let $\tilde{C}$ be the normalization of $C \times_B \tilde{B}$, and $\gamma_{\tilde{C}} \in \operatorname{Aut} \tilde{C}$ the lift of $\gamma_{\tilde{B}}$. Let $\tilde{C}$ be the (hyper-elliptic) involution corresponding to $\tilde{C} \to \tilde{B}$.

Let F be an elliptic curve, and $F[2]$ be the group of 2-torsion points of F. Let $S = (\tilde{C} \times F)/\langle \gamma_{\tilde{C}} \times -1_F \rangle$. Then one checks easily as in Example 4.4 that the group G generated by $\overline{\tau_{\tilde{C}} \times -1_F}$ and $\{\overline{\mathrm{id}_{\tilde{C}} \times t_a} \mid a \in F[2]\}$ is isomorphic to $\mathbb{Z}_2^{\oplus 3}$ and acts trivially on $H^2(S, \mathbb{Q})$.

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